Jacobian and Hessian Matrices

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First Order Partial Derivatives

- The problem with functions of more than one variable is that there is more than one variable.
- Concentrate on changing one of the variables at a time, while the remaining variable(s) are held fixed.
Example 1

- Find the first order derivative of this with respect to $y$ and then $x$.

$$f(x, y) = 2x^2 y^3$$
Solution 1
Second Order Partial Derivatives

- Consider the case of a function of two variables, since both of the first order partial derivatives are also functions of \( x \) and \( y \) we could in turn differentiate each with respect to \( x \) or \( y \).

- This means that for the case of a function of two variables there will be a total of four possible second order derivatives. Here they are and the notations that we’ll use to denote them.

\[
(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}
\]

\[
(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x}
\]

\[
(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}
\]

\[
(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}
\]
Example 2

\[ f(x, y) = xe^{-x^2y^2} \]
Solution 2
Jacobian Matrices

- The matrix of all first-order partial derivatives of a vector- or scalar-valued function with respect to another vector
- The Jacobian of a function describes the orientation of a tangent plane to the function at a given point.
- Likewise, the Jacobian can also be thought of as describing the amount of "stretching" that a transformation imposes.
More generally if $f : \mathbb{R}^m \rightarrow \mathbb{R}$ then we take the derivative at $p$ to be the row vector

$$
\left( \frac{\partial f}{\partial x_1} (p), \frac{\partial f}{\partial x_2} (p), \ldots, \frac{\partial f}{\partial x_m} (p) \right) = \nabla_p f
$$

Now take $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ where $f$ is as in equation (1), then the natural candidate for the derivative of $f$ at $p$ is

$$
J_p f = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \ldots & \frac{\partial f_1}{\partial x_m} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \ldots & \frac{\partial f_2}{\partial x_m} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \ldots & \frac{\partial f_n}{\partial x_m}
\end{pmatrix}
$$
One way of visualizing $f$, say, $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is to think of $f$ as a transformation between co-ordinate planes.
Example 3

Find the Jacobian matrix of this system at (1, 2, 3)

\[ f_1 = x + y + z \]
\[ f_2 = xyz \]
Solution 3
The Jacobian determinant at a given point gives important information about the behavior of $F$ near that point.

For instance, the continuously differentiable function $F$ is invertible near a point $p$ if the Jacobian determinant at $p$ is non-zero.

- This is the inverse function theorem.

Furthermore, if the Jacobian determinant at $p$ is positive, then $F$ preserves orientation near $p$.

- If it is negative, $F$ reverses orientation.

The absolute value of the Jacobian determinant at $p$ gives us the factor by which the function $F$ expands or shrinks volumes near $p$. 
Example 4

Find the Jacobian matrix and determinant of this system at \((3, 6)\)

\[ f_1 = x^2 y + 2x^3 y + y^4 \]
\[ f_2 = y^3 + y \sin x + 2xy \]
Solution 4
Hessian Matrices

- The square matrix of second-order partial derivatives of a function
- Describes the local curvature of a function of many variables
- If all second partial derivatives of $f$ exist, then the Hessian matrix of $f$ is the matrix

$$ H(f) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{bmatrix}. $$
Example 5

- Find the Hessian matrix of the function at (2, 3)

\[ f(x, y) = 3x^4y + \cos y \]
Solution 5
Hessian Determinant

- If the gradient of $f$ (i.e., its derivative in the vector sense) is zero at some point $x$, then $f$ has a critical point (or stationary point) at $x$.
- The determinant of the Hessian at $x$ is then called the discriminant.
- If this determinant is zero then $x$ is called a degenerate critical point of $f$, this is also called a non-Morse critical point of $f$.
- Otherwise it is non-degenerate, this is called a Morse critical point of $f$. 
Example 6

- Find the Hessian matrix and the determinant

\[ f(x, y) = x^4 + e^x + 2xy^2 \]
Solution 6
THE END
3) \( z = x^2 + 2xy + y^3 \) \\
\[
\text{Jacobian } \frac{d}{dx} x^2 + 2xy + y^3 = 2x + 2y \\
\frac{d}{dy} = 2x + 3y^2 \\
\frac{d}{dx} = \cos x \\
\frac{d}{dy} = 1 \\
\begin{bmatrix}
12 & 5x \\
\cos x & 1
\end{bmatrix} = \begin{bmatrix}
2x + 2y & 2x + 3y^2 \\
\cos x & 1
\end{bmatrix}
4) \[
\begin{bmatrix}
12 & 52 \\
\cos 2 & 1
\end{bmatrix}
\]

\[= 12 - 52 \cos 2\]
(6) \[ f_x = \frac{8}{3} x^3 + 12xy \]
\[ f_y = -\sin y + 6x^2 \]
\[ f_{xy} = 8x^2 + 2y \]
\[ f_{yy} = -\cos y \]

\[
\begin{bmatrix}
8x^2 + 12y & 12x \\
12x & -\cos y
\end{bmatrix} = \begin{bmatrix}
12\pi & 0 \\
0 & 1
\end{bmatrix}
\]

\[ T \approx 12\pi \]